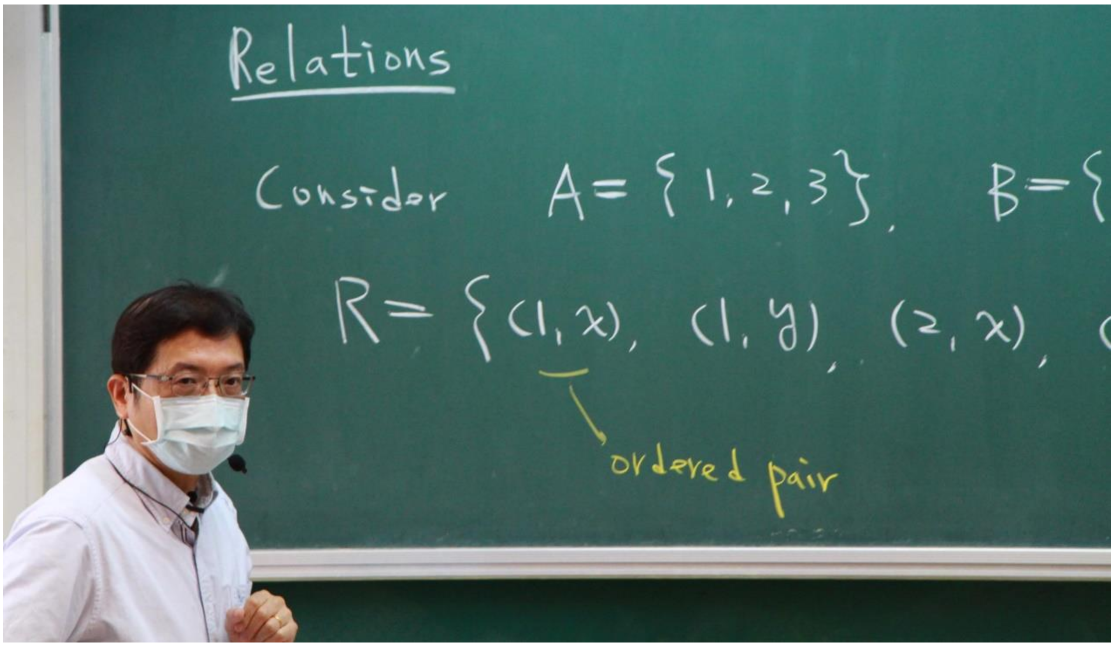
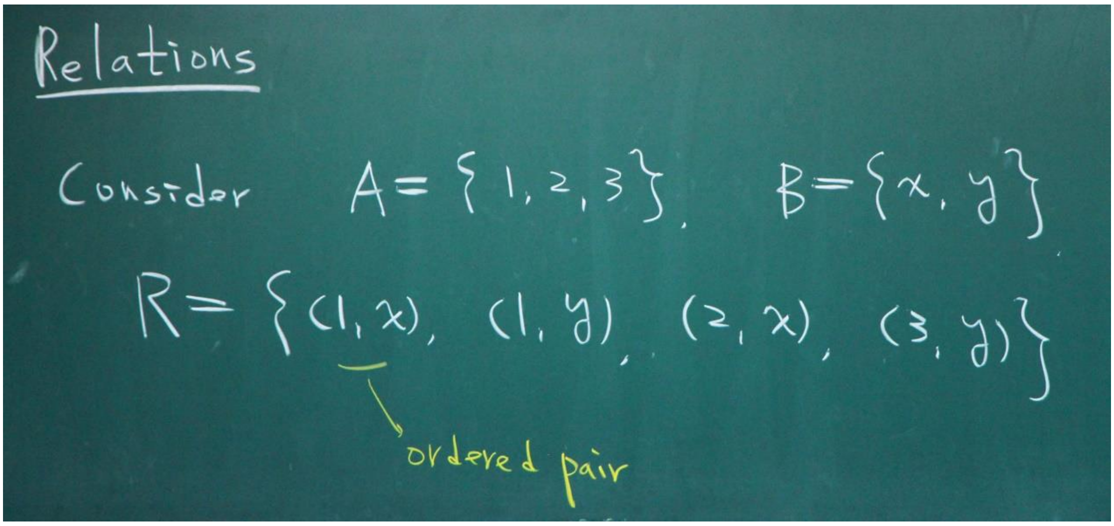
【10920 趙啟超教授離散數學 / 第 6 堂版書】



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ordered pair

$A \times B = \{(a, b) : a \in A, b \in B\}$

Cartesian product

**Def** A (binary) relation $R$ from $A$ to $B$ is a subset of $A \times B$.

## Relations

Consider $A = \{1, 2, 3\}$, $B = \{x, y\}$.

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A relation from $A$ to $B$, where $|A| = m$ and $|B| = n$, can be represented by an $m \times n$ zero-one relation matrix.

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$$M(R) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

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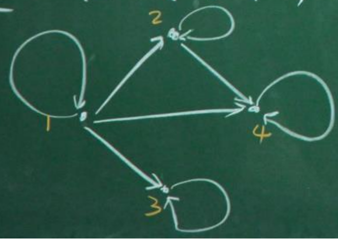


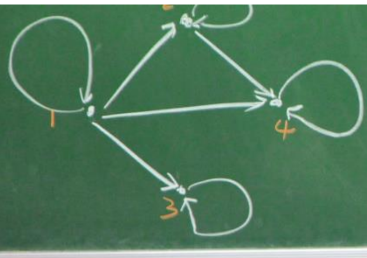


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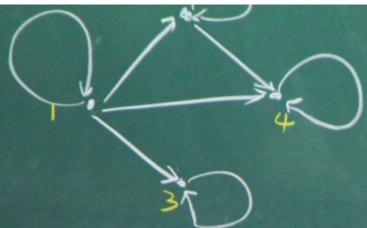


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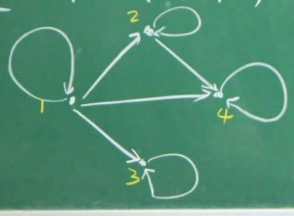


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## Equivalence Relations

**Def** A relation $R$ on a set $A$ is an equivalence relation if the following conditions hold:

1. $a R a$ for all $a \in A$. (reflexive)
2. $a R b \Rightarrow b R a$ (symmetric)
3. $a R b$ and $b R c \Rightarrow a R c$ (transitive)

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Example $A = \mathbb{Z}$, the set of all integers

$aRb \Leftrightarrow a-b = 3k$ for some $k \in \mathbb{Z}$
( or $3 \mid a-b$ )

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3. Transitive: $a R b$ and $b R c$

$$\Rightarrow a-b = 3k_1 \text{ and } b-c = 3k_2$$

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$a \equiv b \pmod{m}$ "a is congruent to b modulo m."

if $a-b = mk$ for some integer $k$

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$$(a_n\cdot 10^n + a_{n-1}\cdot 10^{n-1} + \cdots + a_1\cdot 10 + a_0) \bmod 3$$

$$= \big((a_n\cdot 10^n) \bmod 3 + (a_{n-1}\cdot 10^{n-1}) \bmod 3 + \cdots + (a_1\cdot 10) \bmod 3 + a_0 \bmod 3\big) \bmod 3$$

$$= \big(a_n\cdot(10^n \bmod 3) + a_{n-1}\cdot(10^{n-1} \bmod 3) + \cdots + a_1\cdot(10 \bmod 3) + a_0\big) \bmod 3$$

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$\therefore$ R is an equivalence relation.

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$$= (a_n + a_{n-1} + \cdots + a_1 + a_0) \bmod 3$$

since $10^n \bmod 3 = 1$ for $n \geq 0$.